Generalizations of Hölder inequality,

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The inequality of Hölder experienced over time more proofs and generalizations. The most proof is therefore based on means inequality. Using this idea and the results from article "A generalization of means inequality" of the author, we give some generalizations of Hölder's inequality, and in some situations we obtain known results that are Callabaut, Mitrinović and Pečarić.

Proposition 1. *If* p_1, p_2, a_1, a_2 *are positive real number so that* $a_1^{p_1} \cdot a_2^{p_2} = 1, a_1 \neq a_2$ *then the function* $f : [0,\infty) \to \mathbb{R}, f(x) = p_1 a_1^x + p_2 a_2^x$ *is strict increasing..*

Proof. If we put $a_1 = a > 1$, then the function can be write $f(x) = p_2\left(ta^x + \frac{1}{a^{tx}}\right)$, where $t = \frac{p_1}{p_2}$. Because $f'(x) = p_2 t \cdot \frac{a^{(t+1)x} - 1}{a^{tx}} \cdot \ln a > 0$, result that the function is strict increasing ■

Theorem 1. *If* $\alpha, \beta \in (1,\infty), \frac{1}{\alpha} + \frac{1}{\beta} = 1, x \in [0,1], p_i, a_i, b_i \in (0,\infty), A = \sum_{i=1}^n p_i a_i^\alpha$, $B = \sum_{i=1}^n p_i b_i^\beta, A_i = \frac{a_i^\alpha}{A}, B_i = \frac{b_i^\beta}{B}, i = \overline{1,n}$, *then:*

a) holds the inequalities

$$\left(\sum_{i=1}^n p_i a_i^\alpha\right)^{1/\alpha}\left(\sum_{i=1}^n p_i b_i^\beta\right)^{1/\beta} \geq \frac{1}{\alpha}\left(\frac{B}{A}\right)^{x/\beta}\sum_{i=1}^n p_i a_i^{\alpha x}(a_i b_i)^{1-x} + \frac{1}{\beta}\left(\frac{A}{B}\right)^{x/\alpha}\sum_{i=1}^n p_i b_i^{\beta x}(a_i b_i)^{1-x} \geq$$

$$\geq \left(\sum_{i=1}^n p_i a_i^{\alpha x}(a_i b_i)^{1-x}\right)^{1/\alpha}\left(\sum_{i=1}^n p_i b_i^{\beta x}(a_i b_i)^{1-x}\right)^{1/\beta} \geq \sum_{i=1}^n a_i b_i\ ;$$

b) holds the inequality

$$\left(\sum_{i=1}^n p_i a_i^\alpha\right)^{1/\alpha}\left(\sum_{i=1}^n p_i b_i^\beta\right)^{1/\beta} \geq \left[\frac{1}{\alpha A^x}\sum_{i=1}^n p_i a_i^{\alpha x}(a_i b_i)^{1-x} + \frac{1}{\beta B^x}\sum_{i=1}^n p_i b_i^{\beta x}(a_i b_i)^{1-x}\right]^{1/(1-x)};$$

c) the functions $f, g : \mathbb{R} \to (0,\infty)$,

$$f(x) = \frac{1}{\alpha}\left(\frac{B}{A}\right)^{x/\beta}\sum_{i=1}^n p_i a_i^{\alpha x}(a_i b_i)^{1-x} + \frac{1}{\beta}\left(\frac{A}{B}\right)^{x/\alpha}\sum_{i=1}^n p_i b_i^{\beta x}(a_i b_i)^{1-x},$$

$$g\left(x\right)=\left(\sum_{i=1}^{n} p_i a_i^{\alpha x}\left(a_i b_i\right)^{1-x}\right)^{1/\alpha}\left(\sum_{i=1}^{n} p_i b_i^{\beta x}\left(a_i b_i\right)^{1-x}\right)^{1/\beta},$$

is strict decreasing on $\left(-\infty,0\right]$ *and strict increasing on* $\left[0,\infty\right)$, *if n-tuples* $\left(a_1^{\alpha},a_2^{\alpha},...,a_n^{\alpha}\right),\left(b_1^{\beta},b_2^{\beta},...,b_n^{\beta}\right)$ *is not proportional;*

d) the function $h:\mathbb{R}-\{1\}\rightarrow\left(0,\infty\right)$,

$$h\left(x\right)=\left[\frac{1}{\alpha A^x}\sum_{i=1}^{n} p_i a_i^{\alpha x}\left(a_i b_i\right)^{1-x}+\frac{1}{\beta B^x}\sum_{i=1}^{n} p_i b_i^{\beta x}\left(a_i b_i\right)^{1-x}\right]^{1/(1-x)},$$

can be extended by continuity on $\overline{\mathbb{R}}$, *and is strict decreasing on* $\overline{\mathbb{R}}$, *if n-tuples* $\left(a_1^{\alpha},a_2^{\alpha},...,a_n^{\alpha}\right),\left(b_1^{\beta},b_2^{\beta},...,b_n^{\beta}\right)$ *is not proportional;*

e) holds the inequalities

$$1\leq\frac{\left(\sum_{i=1}^{n} p_i a_i^{\alpha}\right)^{1/\alpha}\left(\sum_{i=1}^{n} p_i b_i^{\beta}\right)^{1/\beta}}{\sum_{i=1}^{n} p_i a_i b_i}\leq\prod_{i=1}^{n}\left(\frac{a_i^{\alpha}}{b_i^{\beta}}\right)^{\left(p_i/\alpha\beta\right)\left(A_i-B_i\right)},$$

where $A_i=\dfrac{a_i^{\alpha}}{A}, B_i=\dfrac{b_i^{\beta}}{B}$, *the equalities holds if and only if n-tuples* $\left(a_1^{\alpha},a_2^{\alpha},...,a_n^{\alpha}\right)$, $\left(b_1^{\beta},b_2^{\beta},...,b_n^{\beta}\right)$ *are proportional.*

Proof. In according with the previous proposition, the function $f:\left[0,\infty\right)\rightarrow\mathbb{R}$,

$$f\left(x\right)=\frac{1}{\alpha}\left(\frac{a}{a^{1/\alpha}b^{1/\beta}}\right)^{x}+\frac{1}{\beta}\left(\frac{b}{a^{1/\alpha}b^{1/\beta}}\right)^{x}$$

is strict increasing if $\alpha,\beta\in\left(1,\infty\right),\frac{1}{\alpha}+\frac{1}{\beta}=1, a,b\in\left(0,\infty\right) a\neq b.$

If $1\geq x\geq 0$, the inequality $f\left(1\right)\geq f\left(x\right)$ hold and it can be write:

$$\frac{1}{\alpha}a+\frac{1}{\beta}b\geq\frac{1}{\alpha}a^{x}\left(a^{1/\alpha}b^{1/\beta}\right)^{1-x}+\frac{1}{\beta}b^{x}\left(a^{1/\alpha}b^{1/\beta}\right)^{1-x}.$$

If in the last inequality we put $a=\dfrac{a_i^{\alpha}}{A}, b=\dfrac{b_i^{\beta}}{B}$, we multiply with p_i and apply the sum for i from 1 to n we obtain

$$1\geq\frac{1}{\alpha}\frac{\sum_{i=1}^{n} p_i a_i^{\alpha x}\left(a_i b_i\right)^{1-x}}{A^{x}\left(A^{1/\alpha}B^{1/\beta}\right)^{1-x}}+\frac{1}{\beta}\frac{\sum_{i=1}^{n} p_i b_i^{\beta x}\left(a_i b_i\right)^{1-x}}{B^{x}\left(A^{1/\alpha}B^{1/\beta}\right)^{1-x}}.\quad (*)$$

a) If in (*) we multiply with $A^{1/\alpha}B^{1/\beta}$ we obtain the first inequality. For the second inequality we apply the means inequality $\frac{1}{\alpha}p+\frac{1}{\beta}q\geq p^{1/\alpha}q^{1/\beta}$, for

$$p=\left(\frac{B}{A}\right)^{x/\beta}\sum_{i=1}^{n}p_i a_i^{\alpha x}\left(a_i b_i\right)^{1-x},\ q=\left(\frac{A}{B}\right)^{x/\alpha}\sum_{i=1}^{n}p_i b_i^{\beta x}\left(a_i b_i\right)^{1-x}.$$

The last inequality results from point c).

b) If in (*) we multiply with $\left(A^{1/\alpha}B^{1/\beta}\right)^{1-x}$ we obtain the desired think, for $x\neq 1$. The fact that the inequality hold for $x=1$ result from d).

c) The function f is the sum of functions

$$f_i:\mathbb{R}\to(0,\infty),\ f_i(x)=\frac{1}{\alpha}a^x\left(a^{1/\alpha}b^{1/\beta}\right)^{1-x}+\frac{1}{\beta}b^x\left(a^{1/\alpha}b^{1/\beta}\right)^{1-x},i=\overline{1,n},$$

where $a=\frac{a_i^\alpha}{A},b=\frac{b_i^\beta}{B}$, multiply with, respectively $p_i A^{1/\alpha}B^{1/\beta}$. Hence is enough to prove that the functions f_i is strict decreasing on $(-\infty,0]$ and strict increasing on $[0,\infty)$.

Because $f_i'(x)=(1/\alpha\beta)\left(a^{1/\alpha}b^{1/\beta}\right)^{1-x}\left(a^x-b^x\right)(\ln a-\ln b)$ the conclusion is immediately.

The derivative of function g is

$$f'(x)=\alpha\beta f(x)\left(\frac{\sum_{i=1}^{n}p_i a_i^{\alpha x}\left(a_i b_i\right)^{1-x}\ln\left(a_i^\alpha/b_i^\beta\right)}{\sum_{i=1}^{n}p_i a_i^{\alpha x}\left(a_i b_i\right)^{1-x}}-\frac{\sum_{i=1}^{n}p_i b_i^{\beta x}\left(a_i b_i\right)^{1-x}\ln\left(a_i^\alpha/b_i^\beta\right)}{\sum_{i=1}^{n}p_i b_i^{\beta x}\left(a_i b_i\right)^{1-x}}\right).$$

From the article “A generalization of Chebyshev inequalities” we know that if n-tuples $\left(a_1,a_2,...,a_n\right),\left(\frac{b_1}{c_1},\frac{b_2}{c_2},...,\frac{b_n}{c_n}\right)$ is in the same ordered then hold the inequality

$$\left(\sum_{i=1}^{n}p_i c_i\right)\left(\sum_{i=1}^{n}p_i b_i a_i\right)\geq\left(\sum_{i=1}^{n}p_i b_i\right)\left(\sum_{i=1}^{n}p_i c_i a_i\right),$$

but if the previous n-tuples is oppositely ordered then the inequality hold in opposite sense.

Moreover, the equality holds, if and only if the at least one of the two n-tuples have all components equals.

In our situation, because $\ln\left(a_i^\alpha/b_i^\beta\right)$ and $\left(a_i^{\alpha t}\left(a_i b_i\right)^{1-x}\right)\big/\left(b_i^{\beta t}\left(a_i b_i\right)^{1-x}\right)=\left(a_i^\alpha/b_i^\beta\right)^x$ is in the same ordered for $x\geq 0$ and oppositely ordered for $x\leq 0,$ comes the conclusion.

d) Is enough to prove that the function h is a weighted means with order $1-x$.
Because we can write

$$h(x)=\left(\frac{1}{\alpha}\sum_{i=1}^{n}p_i A_i\left(\frac{a_i b_i}{a_i^\alpha}A\right)^{1-x}+\frac{1}{\alpha}\sum_{i=1}^{n}p_i B_i\left(\frac{a_i b_i}{b_i^\beta}B\right)^{1-x}\right)^{1/(1-x)},$$

this means that the function h is a weighted means with $1-x$ order and the weights $\frac{A_1}{\alpha},...,\frac{A_n}{\alpha},\frac{B_1}{\beta},...,\frac{B_n}{\beta},$ so is strict increasing in $1-x,$ so is strict decreasing in $x,$ if the next equalities not holds

$$\frac{a_1b_1}{a_1^{\alpha}}A=...=\frac{a_nb_n}{a_n^{\alpha}}A=\frac{a_1b_1}{b_1^{\beta}}B=...=\frac{a_nb_n}{b_n^{\beta}}B,$$

so if n-tuples $\left(a_1^{\alpha},a_2^{\alpha},...,a_n^{\alpha}\right),\left(b_1^{\beta},b_2^{\beta},...,b_n^{\beta}\right)$ are not proportioned.

e) The inequalities result from

$$A^{1/\alpha}B^{1/\beta}\geq h(0)=\sum_{i=1}^{n}p_ia_ib_i\geq h(1)=A^{1/\alpha}B^{1/\beta}\prod_{i=1}^{n}\left(\frac{a_i^{\alpha}}{b_i^{\beta}}\right)^{(p_i/\alpha\beta)(B_i-A_i)} \quad ■$$

Remark 1. *Because the functions* f *and* g *are decreasing on* $(-\infty,0]$ *and increasing on* $[0,\infty)$, *result that* $f(0)$ *respectively* $g(0)$, whose value *is* $\sum_{i=1}^{n}p_ia_ib_i$, *represent an absolutely minima for this functions.*

Remark 2. *From* $f(1)\geq f(0)$ *but and from* $g(1)\geq g(0)$ *we obtain the inequality of Hölder.*

Remark 3. *The result from c) about g function, belongs to Mitrinović-Pečarić, and represents a generalization of Callabaut for the case* $\alpha=\beta=2$.

Remark 4. *For* $\alpha=\beta=2$ *we obtain the generalizations of Cauchy- Schwartz inequalities. The results for* g *function belong to Callabaut.*

Proposition 2. *If* $p_i,a_i,i=\overline{1,n}$, *are positive real number so that* $\prod_{i=1}^{n}a_i^{p_i}=1$, *then the function* $f:[0,\infty)\to\mathbb{R},\ f(x)=\sum_{i=1}^{n}p_ia_i^x$ *is strict increasing, if* $a_i,i=\overline{1,n}$ *are not all equals.*

Proof. By mathematical induction for $n\geq 2$. First we observe that we can assume $p_1+p_2+...+p_n=1$, else we consider the function $f/(p_1+p_2+...+p_n)$. For $n=2$ we obtain the Proposition 1. Consider the affirmation true for n and we prove for $n+1$.
Be the function $g:[0,\infty)\to\mathbb{R},\ g(x)=\sum_{i=1}^{n+1}p_ia_i^x$, where $\prod_{i=1}^{n+1}a_i^{p_i}=1$. If $a_{n+1}=1$ the problem is closed. If $a_{n+1}<1$, we have

$$g(x)=\sum_{i=1}^{n}p_ia_i^x-\left(\prod_{i=1}^{n}a_i^{p_i}\right)^x+p_{n+1}a_{n+1}^x+\left(\prod_{i=1}^{n}a_i^{p_i}\right)^x=$$

$$=\left(\prod_{i=1}^{n}a_i^{p_i}\right)^x\left(\sum_{i=1}^{n}p_i\left(a_i^{-1}\prod_{i=1}^{n}a_i^{p_i}\right)^{-x}-1\right)+p_{n+1}a_{n+1}^x+a_{n+1}^{-p_{n+1}x},$$

and the problem is closed in this case. If $a_{n+1}>1$,

$$g(x)=\sum_{i=1}^{n}p_ia_i^x+p_{n+1}\left(\prod_{i=1}^{n}a_i^{p_i}\right)^{-x}=\left(\prod_{i=1}^{n}a_i^{p_i}\right)^{-x}\left(p_{n+1}+\sum_{i=1}^{n}\left(a_i^{p_i}\prod_{i=1}^{n}a_i^{p_i}\right)^x\right)$$

and the induction is completed ■

Using the Proposition 2 we can prove like in Theorem 1, the next results, representing the transition from two n-tuples at more.

Theorem 2. *If* $r_j \in (1,\infty), \sum_{j=1}^{m} r_j = 1, p_i, a_{ij} \in (0,\infty), x \in [0,1], A_j = \sum_{i=1}^{n} p_i a_{ij}^{r_j}, j = \overline{1,m},$ $i = \overline{1,n},$ *then:*

a) holds the inequalities

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ij}^{r_j}\right)^{1/r_j} \geq \sum_{j=1}^{m}\left[\frac{1}{r_j}\left(\frac{1}{A_j}\prod_{k=1}^{m} A_k^{1/r_k}\right)^{x}\sum_{i=1}^{n} p_i a_{ij}^{r_j x}\left(\prod_{k=1}^{m} a_{ik}\right)^{1-x}\right] \geq$$

$$\geq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ij}^{r_j x}\left(\prod_{k=1}^{m} a_{ik}\right)^{1-x}\right)^{1/r_j} \geq \sum_{i=1}^{n} p_i \prod_{k=1}^{m} a_{ik}\ ;$$

b) holds the inequality

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ij}^{r_j}\right)^{1/r_j} \geq \left[\sum_{j=1}^{m}\left(\frac{1}{r_j A_j^{x}}\sum_{i=1}^{n} p_i a_{ij}^{r_j x}\left(\prod_{k=1}^{m} a_{ik}\right)^{1-x}\right)\right]^{1/(1-x)}\ ;$$

c) the functions $f, g : \mathbb{R} \to (0,\infty),$

$$f(x) = \sum_{j=1}^{m}\left[\frac{1}{r_j}\left(\frac{1}{A_j}\prod_{k=1}^{m} A_k^{1/r_k}\right)^{x}\sum_{i=1}^{n} p_i a_{ij}^{r_j x}\left(\prod_{k=1}^{m} a_{ik}\right)^{1-x}\right],$$

$$g(x) = \prod_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ij}^{r_j x}\left(\prod_{k=1}^{m} a_{ik}\right)^{1-x}\right)^{1/r_j},$$

are strict decreasing on $(-\infty,0]$ *and are strict increasing* $[0,\infty),$ *if the n-tuples* $\left(a_{1j}^{r_j}, a_{2j}^{r_j}, \ldots, a_{nj}^{r_j}\right), j = \overline{1,m},$ *are not proportional;*

d) the function $h : \mathbb{R} - \{1\} \to (0,\infty), h(x) = \left[\sum_{j=1}^{m}\left(\frac{1}{r_j A_j^{x}}\sum_{i=1}^{n} p_i a_{ij}^{r_j x}\left(\prod_{k=1}^{m} a_{ik}\right)^{1-x}\right)\right]^{1/(1-x)},$

can be extended by continuity on $\overline{\mathbb{R}},$ *and is strict decreasing on* $\overline{\mathbb{R}},$ *if the n-tuples* $\left(a_{1j}^{r_j}, a_{2j}^{r_j}, \ldots, a_{nj}^{r_j}\right), j = \overline{1,m},$ *are not proportional;*

e) holds the inequalities

$$1 \leq \frac{\prod_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ij}^{r_j}\right)^{1/r_j}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{m} p_i a_{ij}\right)} \leq \exp\left(\sum_{j=1}^{m}\left(\frac{1}{r_j A_j}\sum_{i=1}^{n} p_i a_{ij}^{r_j}\ln\frac{a_{ij}^{r_j}}{\prod_{k=1}^{m} a_{ik}}\right)\right).$$

The equalities holds if and only if n-tuples $\left(a_{1j}^{r_j}, a_{2j}^{r_j}, \ldots, a_{nj}^{r_j}\right), j = \overline{1,m},$ *are proportional.*

Theorem 3. *Let* $H:(0,\infty)\times J\to(0,\infty)$, J *interval, a strict monotonous multiplicatively kernel with the next properties:*
1) H *is increasing in the first variable;*
2) H *is derivable in the second variable;*
3) there is $x_0\in J$ *so that* $H(a,x_0)=a,\forall a>0.$
If $\alpha,\beta\in(1,\infty),\frac{1}{\alpha}+\frac{1}{\beta}=1,\ p_i,a_i,b_i\in(0,\infty),\forall i=\overline{1,n},$ *and* $A,B,A_i,B_i,i=\overline{1,n},$ *are like in Theorem 1, then:*
a) $\forall x\in J,x_0\geq x$ *holds the inequalities:*

$$A^{1/\alpha}B^{1/\beta}\geq\frac{1}{\alpha}\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(A_i,x)}{H(B_i,x)}\right)^{1/\beta}+\frac{1}{\beta}\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(B_i,x)}{H(A_i,x)}\right)^{1/\alpha}\geq$$

$$\geq\left(\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(A_i,x)}{H(B_i,x)}\right)^{1/\beta}\right)^{1/\alpha}\left(\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(B_i,x)}{H(A_i,x)}\right)^{1/\alpha}\right)^{1/\beta};$$

b) the function $f:\{x\,|\,x\in J,x_0\geq x\}\to(0,\infty)$,

$$f(x)=\frac{1}{\alpha}\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(A_i,x)}{H(B_i,x)}\right)^{1/\beta}+\frac{1}{\beta}\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(B_i,x)}{H(A_i,x)}\right)^{1/\alpha},$$

is increasing;
c) if for all $x\leq x_0$, *the sequences* $\frac{H(A_i,x)}{H(B_i,x)}$ *and* $\frac{H'(A_i,x)}{H(A_i,x)}-\frac{H'(B_i,x)}{H(B_i,x)},i=\overline{1,n},$ *are in the same ordered, the function* $g:\{x\,|\,x\in J,x_0\geq x\}\to(0,\infty)$

$$g(x)=\left(\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(A_i,x)}{H(B_i,x)}\right)^{1/\beta}\right)^{1/\alpha}\left(\sum_{i=1}^{n}p_ia_ib_i\left(\frac{H(B_i,x)}{H(A_i,x)}\right)^{1/\alpha}\right)^{1/\beta},$$

is increasing.

Proof. From the article “A generalization of means inequality” we know that if H is a multiplicative monotonous kernel like in the hypothesis then the function $h:J\to(0,\infty)$,

$$h(x)=\frac{(1/\alpha)H(a,x)+(1/\beta)H(b,x)}{H^{1/\alpha}(a,x)H^{1/\beta}(b,x)}=\frac{1}{\alpha}\left(\frac{H(a,x)}{H(b,x)}\right)^{1/\beta}+\frac{1}{\beta}\left(\frac{H(b,x)}{H(a,x)}\right)^{1/\alpha},$$

is strict increasing, if $a\neq b$. Thus for $x_0\geq x$ we have $h(x_0)\geq h(x)$ so

$$\frac{1}{\alpha}a+\frac{1}{\beta}b\geq\frac{1}{\alpha}a^{1/\alpha}b^{1/\beta}\left(\frac{H(a,x)}{H(b,x)}\right)^{1/\beta}+\frac{1}{\beta}a^{1/\alpha}b^{1/\beta}\left(\frac{H(b,x)}{H(a,x)}\right)^{1/\alpha}.$$

a) If in the last inequality we put $a=A_i,b=B_i,i=\overline{1,n},$ we multiply with p_i and apply the sum for i from 1 to n we obtain the conclusion.

b) The monotony of function f result from the monotony of function h.

c) About the function g, we have:

$$g'(x)=\frac{1}{\alpha\beta}g(x)\left(\frac{\sum_{i=1}^{n}p_i a_i b_i\left(\frac{H(A_i,x)}{H(B_i,x)}\right)^{1/\beta}\left(\frac{H'(A_i,x)}{H(A_i,x)}-\frac{H'(B_i,x)}{H(B_i,x)}\right)}{\sum_{i=1}^{n}p_i a_i b_i\left(\frac{H(A_i,x)}{H(B_i,x)}\right)^{1/\beta}}-\right.$$

$$\left.-\frac{\sum_{i=1}^{n}p_i a_i b_i\left(\frac{H(B_i,x)}{H(A_i,x)}\right)^{1/\alpha}\left(\frac{H'(A_i,x)}{H(A_i,x)}-\frac{H'(B_i,x)}{H(B_i,x)}\right)}{\sum_{i=1}^{n}p_i a_i b_i\left(\frac{H(B_i,x)}{H(A_i,x)}\right)^{1/\alpha}}\right)\geq 0 \blacksquare$$

Remark 5. *If we consider the kernel* $H:(0,\infty)\times[0,\infty)\to(0,\infty), H(a,x)=a^x$, *then* $x_0=1$, *and if* $0\leq x\leq 1$, *we obtain the results from Theorem 1.*

In the next we present the integral version of previous theorems which can be obtained using the Rieman's sums.

Theorem 4. *If* $\alpha,\beta\in(1,\infty),\frac{1}{\alpha}+\frac{1}{\beta}=1, x\in[0,1], A=\int_a^b p(t)f^{\alpha}(t)dt$, $B=\int_a^b p(t)g^{\beta}(t)dt$, *where* $p, f, g:[a,b]\to(0,\infty)$ *are continuous functions, then:*

a) holds the inequalities:

$$\left(\int_a^b p(t)f^{\alpha}(t)dt\right)^{1/\alpha}\left(\int_a^b p(t)g^{\beta}(t)dt\right)^{1/\beta}\geq\frac{1}{\alpha}\left(\frac{B}{A}\right)^{x/\beta}\int_a^b p(t)f^{\alpha x}(t)\left(f(t)g(t)\right)^{1-x}dt+$$

$$+\frac{1}{\beta}\left(\frac{A}{B}\right)^{x/\alpha}\int_a^b p(t)g^{\beta x}(t)\left(f(t)g(t)\right)^{1-x}dt\geq\left(\int_a^b p(t)f^{\alpha x}(t)\left(f(t)g(t)\right)^{1-x}dt\right)^{1/\alpha}\cdot$$

$$\cdot\left(\int_a^b p(t)g^{\beta x}(t)\left(f(t)g(t)\right)^{1-x}dt\right)^{1/\beta}\geq\int_a^b p(t)f(t)g(t)dt\ ;$$

b) hold the inequality

$$A^{1/\alpha}B^{1/\beta}\geq\left(\frac{1}{\alpha A^x}\int_a^b f^{\alpha x}(t)\left(f(t)g(t)\right)^{1-x}dt+\frac{1}{\beta B^x}\int_a^b g^{\beta x}(t)\left(f(t)g(t)\right)^{1-x}dt\right)^{1/(1-x)};$$

c) the functions $u,v:\mathbb{R}\to(0,\infty)$,

$$u(x)=\frac{1}{\alpha}\left(\frac{B}{A}\right)^{x/\beta}\int_a^b f^{\alpha x}(t)\left(f(t)g(t)\right)^{1-x}dt+\frac{1}{\beta}\left(\frac{A}{B}\right)^{x/\alpha}\int_a^b g^{\beta x}(t)\left(f(t)g(t)\right)^{1-x}dt,$$

$$v(x)=\left(\int_a^b f^{\alpha x}(t)\left(f(t)g(t)\right)^{1-x}dt\right)^{1/\alpha}\left(\int_a^b g^{\beta x}(t)\left(f(t)g(t)\right)^{1-x}dt\right)^{1/\beta},$$

are decreasing on $(-\infty,0]$ *and increasing on* $[0,\infty)$;

d) the function $w:\mathbb{R}-\{1\}\to(0,\infty)$

$$w(x)=\left(\frac{1}{\alpha A^x}\int_a^b f^{\alpha x}(t)\left(f(t)g(t)\right)^{1-x}dt+\frac{1}{\beta B^x}\int_a^b g^{\beta x}(t)\left(f(t)g(t)\right)^{1-x}dt\right)^{1/(1-x)},$$

can be extended by continuity on $\overline{\mathbb{R}}$, *and is increasing on* $\overline{\mathbb{R}}$;

e) hold the inequality

$$1\le\frac{A^{1/\alpha}B^{1/\beta}}{\int_a^b f(t)g(t)dt}\le\exp\left(\frac{1}{\alpha A}\int_a^b p(t)f^{\alpha}(t)\ln\frac{f^{\alpha}(t)}{f(t)g(t)}dt+\right.$$
$$\left.+\frac{1}{\beta B}\int_a^b p(t)g^{\beta}(t)\ln\frac{g^{\beta}(t)}{f(t)g(t)}dt\right).$$

Theorem 5. *If* $r_j\in(1,\infty),\sum_{j=1}^m r_j=1, x\in[0,1], A_j=\int_a^b p(t)f_j^{r_j}(t)dt,$ *where* $p, f_j:[a,b]\to(0,\infty),\ j=\overline{1,m},$ *are continuous functions, then :*

a) holds the inequalities

$$\prod_{j=1}^m A_j^{1/r_j}\ge\sum_{j=1}^m\left[\frac{1}{r_j}\left(\frac{1}{A_j}\prod_{k=1}^m A_k^{1/r_k}\right)^x\int_a^b p(t)f_j^{r_jx}(t)\left(\prod_{k=1}^m f_k(t)\right)^{1-x}dt\right]\ge$$
$$\ge\prod_{j=1}^m\left(\int_a^b p(t)f_j^{r_jx}(t)\left(\prod_{k=1}^m f_k(t)\right)^{1-x}dt\right)^{1/r_j}\ge\int_a^b p(t)\prod_{k=1}^m f_k(t)dt;$$

b) hold the inequality

$$\prod_{j=1}^m\left(\int_a^b p(t)f_j^{r_jx}(t)\right)^{1/r_j}\ge\left[\sum_{j=1}^m\left(\frac{1}{r_jA_j^x}\int_a^b p(t)f_j^{r_jx}(t)\left(\prod_{k=1}^m f_k(t)\right)^{1-x}dt\right)\right]^{1/(1-x)};$$

c) the functions $f,g:\mathbb{R}\to(0,\infty)$,

$$f(x)=\sum_{j=1}^{m}\left[\frac{1}{r_j}\left(\frac{1}{A_j}\prod_{k=1}^{m}A_k^{1/r_k}\right)^{x}\int_a^b p(t)\, f_j^{r_j x}(t)\left(\prod_{k=1}^{m} f_k(t)\right)^{1-x} dt\right],$$

$$g(x)=\prod_{j=1}^{m}\left(\int_a^b p(t)\, f_j^{r_j x}(t)\left(\prod_{k=1}^{m} f_k(t)\right)^{1-x} dt\right)^{1/r_j},$$

are decreasing on $(-\infty,0]$ *and increasing on* $[0,\infty)$;

d) the function $h:\mathbb{R}-\{1\}\to(0,\infty)$,

$$h(x)=\left[\sum_{j=1}^{m}\left(\frac{1}{r_j A_j^x}\int_a^b p(t)\, f_j^{r_j x}(t)\left(\prod_{k=1}^{m} f_k(t)\right)^{1-x} dt\right)\right]^{1/(1-x)},$$

can be extended on $\overline{\mathbb{R}}$, *and is increasing on* $\overline{\mathbb{R}}$.

e) holds the inequality

$$1\le\frac{\prod_{j=1}^{m}A_j^{1/r_j}}{\int_a^b\left(\prod_{j=1}^{m}p(t)\, f_j(t)\right)dt}\le\exp\left(\sum_{j=1}^{m}\left(\frac{1}{r_j A_j}\int_a^b p(t)\, f_j(t)\ln\frac{f_j^{r_j}}{\prod_{k=1}^{m} f_k(t)}dt\right)\right);$$

Theorem 6. *Let* $H:(0,\infty)\times J\to(0,\infty)$, J *interval, a strict monotonous multiplicatively kernel with the next properties:*

1) H *is increasing in the first variable;*

2) H *is derivable in the second variable;*

3) there is $x_0\in J$ *so that* $H(a,x_0)=a,\forall a>0.$

If $\alpha,\beta\in(1,\infty),\dfrac{1}{\alpha}+\dfrac{1}{\beta}=1, p_i,a_i,b_i\in(0,\infty),\forall i=\overline{1,n},$ *and* $A,B,$ *are like in Theorem 4, then:*

a) $\forall x\in J, x_0\ge x$ *holds the inequalities*

$$A^{1/\alpha}B^{1/\beta}\ge\frac{1}{\alpha}\int_a^b p(t)\, f(t)\, g(t)\left(\frac{H\left(A^{-1}f^{\alpha}(t),x\right)}{H\left(B^{-1}g^{\beta}(t),x\right)}\right)^{1/\beta}dt+$$

$$+\frac{1}{\beta}\int_a^b p(t)\, f(t)\, g(t)\left(\frac{H\left(B^{-1}g^{\beta}(t),x\right)}{H\left(A^{-1}f^{\alpha}(t),x\right)}\right)^{1/\alpha}dt\ge$$

$$\ge\left(\int_a^b p(t)\, f(t)\, g(t)\left(\frac{H\left(A^{-1}f^{\alpha}(t),x\right)}{H\left(B^{-1}g^{\beta}(t),x\right)}\right)^{1/\beta}dt\right)^{1/\alpha}.$$

$$\cdot\left(\int_a^b p(t)\,f(t)\,g(t)\left(\frac{H\left(B^{-1}g^{\beta}(t),x\right)}{H\left(A^{-1}f^{\alpha}(t),x\right)}\right)^{1/\alpha}dt\right)^{1/\beta};$$

b) the function $f:\{x\,|\,x\in J, x_0\ge x\}\to(0,\infty)$,

$$f(x)=\frac{1}{\alpha}\int_a^b p(t)\,f(t)\,g(t)\left(\frac{H\left(A^{-1}f^{\alpha}(t),x\right)}{H\left(B^{-1}g^{\beta}(t),x\right)}\right)^{1/\beta}dt+$$

$$+\frac{1}{\beta}\int_a^b p(t)\,f(t)\,g(t)\left(\frac{H\left(B^{-1}g^{\beta}(t),x\right)}{H\left(A^{-1}f^{\alpha}(t),x\right)}\right)^{1/\alpha}dt,$$

is increasing;

c) if the functions $\dfrac{H\left(A^{-1}f^{\alpha},x\right)}{H\left(B^{-1}g^{\beta},x\right)}$ *and* $\dfrac{H'\left(A^{-1}f^{\alpha},x\right)}{H\left(A^{-1}f^{\alpha},x\right)}-\dfrac{H'\left(B^{-1}g^{\beta},x\right)}{H\left(B^{-1}g^{\beta},x\right)}, i=\overline{1,n},$ *are similar monotonous* $\forall x\le x_0$*, the function* $g:\{x\,|\,x\in J, x_0\ge x\}\to(0,\infty)$

$$g(x)=\left(\int_a^b p(t)\,f(t)\,g(t)\left(\frac{H\left(A^{-1}f^{\alpha}(t),x\right)}{H\left(B^{-1}g^{\beta}(t),x\right)}\right)^{1/\beta}dt\right)^{1/\alpha}\cdot$$

$$\cdot\left(\int_a^b p(t)\,f(t)\,g(t)\left(\frac{H\left(B^{-1}g^{\beta}(t),x\right)}{H\left(A^{-1}f^{\alpha}(t),x\right)}\right)^{1/\alpha}dt\right)^{1/\beta},$$

is increasing.

In the next, using the inequalities from point e) of Theorem 1 we give some generalizations of Minkovski inequalitie.

Theorem 7. *If* $\alpha,\beta\in(1,\infty), \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1, p_i, a_i, b_i\in(0,\infty), \forall i=\overline{1,n}, A=\sum_{i=1}^n p_i a_i^{\alpha},$

$B=\sum_{i=1}^n p_i b_i^{\alpha}, A_i=\dfrac{a_i^{\alpha}}{A}, B_i=\dfrac{b_i^{\alpha}}{B}, C_i=\dfrac{(a_i+b_i)^{\alpha}}{\sum_{i=1}^n p_i(a_i+b_i)^{\alpha}}, i=\overline{1,n}, u=\dfrac{A^{1/\alpha}}{A^{1/\alpha}+B^{1/\alpha}},$

$v=\dfrac{B^{1/\alpha}}{A^{1/\alpha}+B^{1/\alpha}}$*, then holds the inequalities*

$$\prod_{i=1}^n\left(\frac{a_i+b_i}{a_i}\right)^{(p_iu/\beta)(C_i-A_i)}\prod_{i=1}^n\left(\frac{a_i+b_i}{b_i}\right)^{(p_iv/\beta)(C_i-B_i)}\le u\prod_{i=1}^n\left(\frac{a_i+b_i}{a_i}\right)^{(p_i/\beta)(C_i-A_i)}+$$

$$+v\prod_{i=1}^{n}\left(\frac{a_i+b_i}{b_i}\right)^{(p_i/\beta)(C_i-B_i)} \le \frac{\left(\sum_{i=1}^{n} p_i\left(a_i+b_i\right)^{\alpha}\right)^{1/\alpha}}{\left(\sum_{i=1}^{n} p_i a_i^{\alpha}\right)^{1/\alpha}+\left(\sum_{i=1}^{n} p_i b_i^{\alpha}\right)^{1/\alpha}} \le 1.$$

The equalities hold if and only if n-tuples $a_i, b_i, i=\overline{1,n}$, *are proportional.*

Proof. The inequality from left is the means inequality and the inequality from right is the inequality of Minkovski. For to prove the inequality from middle we proceed like in in classical proof of Minkovski's inequality by way of Hölder's inequality but in this case we use the inequality from right of point e), Theorem 1.

$$\sum_{i=1}^{n} p_i\left(a_i+b_i\right)^{\alpha}=\sum_{i=1}^{n} p_i a_i\left(a_i+b_i\right)^{\alpha-1}+\sum_{i=1}^{n} p_i b_i\left(a_i+b_i\right)^{\alpha-1} \ge$$

$$\ge A^{1/\alpha}\left(\frac{\left(a_i+b_i\right)^{\alpha}}{C_i}\right)^{1/\beta}\prod_{i=1}^{n}\left(\frac{a_i+b_i}{a_i}\right)^{(p_i/\beta)(C_i-A_i)}+B^{1/\alpha}\left(\frac{\left(a_i+b_i\right)^{\alpha}}{C_i}\right)^{1/\beta}\prod_{i=1}^{n}\left(\frac{a_i+b_i}{b_i}\right)^{(p_i/\beta)(C_i-B_i)}$$

and after a division by $\left(\frac{\left(a_i+b_i\right)^{\alpha}}{C_i}\right)^{1/\beta}=\sum_{i=1}^{n} p_i\left(a_i+b_i\right)^{\alpha}$ we obtain the desired result ■

Theorem 8. *If* $\alpha,\beta\in(1,\infty), \frac{1}{\alpha}+\frac{1}{\beta}=1, p_i, a_{ij}\in(0,\infty), A_{ij}=\frac{a_{ij}^{\alpha}}{\sum_{i=1}^{n} p_i a_{ij}^{\alpha}},$

$C_i=\frac{\left(\sum_{j=1}^{m} a_{ij}\right)^{\alpha}}{\sum_{i=1}^{n} p_i\left(\sum_{j=1}^{m} a_{ij}\right)^{\alpha}}, u_j=\frac{\left(\sum_{i=1}^{n} p_i a_{ij}^{\alpha}\right)^{1/\alpha}}{\sum_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ij}^{\alpha}\right)^{1/\alpha}}, \forall i=\overline{1,n},\ j=\overline{1,m},$ *then holds the inequalities*

$$\exp\left(\frac{1}{\beta}\sum_{j=1}^{m} u_j\sum_{i=1}^{n} p_i\left(C_i-A_{ij}\right)\ln\frac{\sum_{j=1}^{m} a_{ij}}{a_{ij}}\right)\le\sum_{j=1}^{m} u_j\exp\left(\frac{1}{\beta}\sum_{i=1}^{n} p_i\left(C_i-A_{ij}\right)\ln\frac{\sum_{j=1}^{m} a_{ij}}{a_{ij}}\right)\le$$

$$\le\frac{\left(\sum_{i=1}^{n} p_i\left(\sum_{j=1}^{m} a_{ij}\right)^{\alpha}\right)^{1/\alpha}}{\sum_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ij}^{\alpha}\right)^{1/\alpha}}\le 1.$$

The equalities holds if and only if n-tuples $\left(a_{1j}, a_{2j}, \ldots, a_{nj}\right), j=\overline{1,m}$, *are proportional.*

Theorem 9. *If* $\alpha,\beta\in(1,\infty), \frac{1}{\alpha}+\frac{1}{\beta}=1, A=\int_a^b p(t) f^{\alpha}(t)\,dt, B=\int_a^b p(t) g^{\alpha}(t)\,dt,$

where $p, f, g:[a,b]\to(0,\infty)$ *are continuous functions, then holds the inequalities*

$$\exp\left(\frac{u}{\beta}\int_a^b p(t)\left(\frac{1}{C}\left(f(t)+g(t)\right)^{\alpha}-\frac{1}{A}f^{\alpha}(t)\right)\ln\frac{f(t)+g(t)}{f(t)}\,dt+\right.$$

$$+\frac{v}{\beta}\int_a^b p(t)\left(\frac{1}{C}\left(f(t)+g(t)\right)^{\alpha}-\frac{1}{B}g^{\alpha}(t)\right)\ln\frac{f(t)+g(t)}{g(t)}dt\Bigg)\le$$

$$u\exp\left(\frac{1}{\beta}\int_a^b p(t)\left(\frac{1}{C}\left(f(t)+g(t)\right)^{\alpha}-\frac{1}{A}f^{\alpha}(t)\right)\ln\frac{f(t)+g(t)}{f(t)}dt\right)+$$

$$+v\exp\left(\frac{1}{\beta}\int_a^b p(t)\left(\frac{1}{C}\left(f(t)+g(t)\right)^{\alpha}-\frac{1}{B}g^{\alpha}(t)\right)\ln\frac{f(t)+g(t)}{g(t)}dt\right)\le$$

$$\le\frac{\left(\int_a^b p(t)\left(f(t)+g(t)\right)^{\alpha}dt\right)^{1/\alpha}}{\left(\int_a^b p(t)f^{\alpha}(t)dt\right)^{1/\alpha}+\left(\int_a^b p(t)g^{\alpha}(t)dt\right)^{1/\alpha}}\le 1,$$

where $u=\frac{A^{1/\alpha}}{A^{1/\alpha}+B^{1/\alpha}}, v=\frac{B^{1/\alpha}}{A^{1/\alpha}+B^{1/\alpha}}, C=\int_a^b p(t)\left(f(t)+g(t)\right)^{\alpha}dt.$

Theorem 10. *If* $\alpha,\beta\in(1,\infty),\frac{1}{\alpha}+\frac{1}{\beta}=1, B_j=\int_a^b p(t)f_j^{\alpha}(t)dt, B=\int_a^b\left(\sum_{j=1}^m p(t)f_j(t)\right)^{\alpha}dt,$

$u_j=\frac{\left(\int_a^b p(t)f_j^{\alpha}(t)dt\right)^{1/\alpha}}{\sum_{j=1}^m\left(\int_a^b p(t)f_j^{\alpha}(t)dt\right)^{1/\alpha}}$ *where* $p, f_j:[a,b]\to(0,\infty),\ j=\overline{1,m},$ *are continuous functions, then holds the inequalities:*

$$\exp\left(\frac{1}{\beta}\sum_{j=1}^m u_j\int_a^b p(t)\left(\frac{1}{B}\left(\sum_{i=1}^m f_i(t)\right)^{\alpha}-\frac{1}{B_j}f_j^{\alpha}(t)\right)\ln\frac{\sum_{i=1}^m f_i(t)}{f_j(t)}dt\right)\le$$

$$\le\sum_{j=1}^m u_j\exp\left(\frac{1}{\beta}\int_a^b p(t)\left(\frac{1}{B}\left(\sum_{i=1}^m f_i(t)\right)^{\alpha}-\frac{1}{B_j}f_j^{\alpha}(t)\right)\ln\frac{\sum_{i=1}^m f_i(t)}{f_j(t)}dt\right)\le$$

$$\le\frac{\left(\int_a^b p(t)\left(\sum_{j=1}^m f_j(t)\right)^{\alpha}dt\right)^{1/\alpha}}{\sum_{j=1}^m\left(\int_a^b p(t)f_j^{\alpha}(t)dt\right)^{1/\alpha}}\le 1.$$

Finally we remark that if $\alpha\in(0,1)$ the inequalities change its sense.

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